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A NOTE ON A THEOREM OF YAMAMURO, (U)
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C. B./Garcia and F. J./Gould

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A NOTE ON A THEOREM OF YAMAMURO

by

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Conditions are presented which are necessary and sufficient for the existence of a fixed point of a C^1 map.

1. Introduction

We consider the classical problem of characterizing the existence of a fixed point of a map G . In 1963 the following extension of the Leray-Schauder Theorem was presented by Yamamuro [1]:

Let $C \subseteq \mathbb{R}^n$ be an open bounded set, and assume that $G: \bar{C} \rightarrow \mathbb{R}^n$ is continuous (\bar{C} denoting the closure of the set C). Suppose there is an $x^0 \in C$ such that $Gx \neq \lambda x + (1 - \lambda)x^0$ whenever $\lambda > 1$ and $x \in \dot{C}$ (\dot{C} denoting the boundary of C). Then G has a fixed point in \bar{C} .

The weakness of this theorem is indicated by the following trivial example. Let

$$G_1(x_1, x_2) = 1$$

$$G_2(x_1, x_2) = 2x_2 - 1.$$

The point $(1, 1)$ is a unique fixed point of G . However, by looking at the cases (i) $x_2^0 \neq 1$, $\lambda \neq 2$, and (ii) $x_2^0 = 1$, $\lambda = 2$, it can be verified that Yamamuro's assumptions fail. That is:

Given any open bounded set $C \subseteq \mathbb{R}^2$ such that $(1, 1) \in \bar{C}$, and given any $x^0 \in C$, then for some $x \in \dot{C}$ and $\lambda > 1$, we have $Gx = \lambda x + (1 - \lambda)x^0$.

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In this paper we shall further restrict (beyond mere continuity) the class of functions to be considered. By so doing it will be shown that Yamamuro's assumptions, applied to this restricted class, are the weakest possible conditions for the existence of a fixed point, i.e., they are necessary and sufficient.

2. The Extension of Yamamuro's Theorem

Our results are obtained by further restricting the function G and then transforming G to a new function H . The new function will have a fixed point if and only if G does, and a fixed point of the new function can be completely characterized. Let $C \subseteq \mathbb{R}^n$ be an open bounded set on which the function $G: \bar{C} \rightarrow \mathbb{R}^n$ is continuously differentiable. Given any such C , let $J_G(x)$ denote the Jacobian matrix of G at the point x , and let us define the subset $S(C)$ as

$$S(C) = \{x \in C: \det(I - J_G(x)) \neq 0\}.$$

We now make the regularity assumption that $S(C)$ is not empty. It is immediate from Sard's Theorem that this will be true if $(I - G)(\bar{C})$ is not a set of measure zero in \mathbb{R}^n . Now given $x^0 \in S(C)$, define $H: \bar{C} \rightarrow \mathbb{R}^n$ as

$$H_{x^0}(x) = x - (I - J_G(x^0))^{-1} (x - G(x)).$$

Note that x^* is a fixed point of G if and only if x^* is a fixed point of H . Consequently Yamamuro's Theorem applied to the function H immediately yields the following result.

Lemma 1. Suppose there exists a set C , as above, and suppose there is an $x^0 \in S(C)$ such that $H_{x^0}(x) \neq \lambda x + (1 - \lambda)x^0$ whenever $\lambda > 1$ and $x \in \bar{C}$. Then H and hence G has a fixed point in \bar{C} .

Let us now say that x^* is a regular fixed point of G if $(I - J_G(x^*))$ is nonsingular. The following theorem demonstrates that, for C^1 functions G with a regular fixed point, although the hypotheses of Yamamuro may fail to hold, they will in fact be satisfied when applied to the function H as in Lemma 1.

Theorem. Suppose G has a regular fixed point x^* and suppose G is continuously differentiable in a neighborhood of x^* . Then there is an open bounded set $C \subseteq \mathbb{R}^n$, with $x^* \in \bar{C}$, and an $x^0 \in S(C)$ such that $H_{x^0}(x) \neq \lambda x + (1 - \lambda)x^0$ whenever $\lambda > 1$ and $x \in \dot{C}$.

Proof: By the assumptions of the theorem, $[I - J_G(x)]^{-1}$ is continuous in an open neighborhood N of x^* . Given any $x \in \mathbb{R}^n$, $x^0 \in N$, suppose that, for some $\lambda > 1$,

$$(1) \quad H_{x^0}(x) = \lambda x + (1 - \lambda)x^0.$$

To simplify notation, let $F(x) = x - G(x)$. By assumption $F(x^*) = 0$. Now expression (1) becomes

$$(2) \quad J_F^{-1}(x^0) F(x) = \eta(x - x^0)$$

for some $\eta < 0$. Then

$$J_F^{-1}(x^0) [F(x^0) + J_F(x^0)(x - x^0) + \underline{\varepsilon}(x, x^0)||x - x^0||] = \eta(x - x^0)$$

where $\underline{\varepsilon}(x, x^0) \rightarrow 0$ as $||x - x^0|| \rightarrow 0$. Thus,

$$(3) \quad J_F^{-1}(x^0) [F(x^0) + \underline{\varepsilon}(x, x^0)||x - x^0||] = (\eta - 1)(x - x^0).$$

Let $B_\delta(x^*)$ and $S_\delta(x^*)$ denote, respectively, the open ball and its boundary (the sphere) of radius δ about x^* . Pick $\delta > 0$ such that $\bar{B}_\delta(x^*) \subset N$. Since J_F^{-1} is continuous on the compact set $\bar{B}_\delta(x^*)$,

we can find a positive number M for which $||J_F^{-1}(x^0)|| \leq M$ for $x^0 \in \bar{B}_\delta(x^*)$.

Suppose $v > 0$ is chosen so that $||x - x^0|| \leq v \Rightarrow ||\underline{\varepsilon}(x, x^0)|| < \frac{1}{2M}$. Then

if $x \in S_{v/2}(x^*)$ and $x^0 \in B_{v/2}(x^*)$ we have $||x - x^0|| \leq v$ and hence

$||\underline{\varepsilon}(x, x^0)|| < \frac{1}{2M}$. Consequently,

$$(4) \quad ||\underline{\varepsilon}(x, x^0)|| ||x - x^0|| < \frac{||x - x^0||}{2M}.$$

Now since $F(x^*) = 0$ there is a positive $\rho < \min\{\delta, v/4\}$ such that

$$(5) \quad x^0 \in B_\rho(x^*) \Rightarrow ||F(x^0)|| < \frac{v/4}{2M}.$$

But if $x \in S_{v/2}(x^*)$ and $\rho < v/4$ then

$$(6) \quad x^0 \in B_\rho(x^*) \Rightarrow v/4 < ||x - x^0||.$$

Hence, $x \in S_{v/2}(x^*)$ and $x^0 \in B_\rho(x^*)$, along with (5) and (6), imply

$$(7) \quad ||F(x^0)|| < \frac{||x - x^0||}{2M}.$$

Hence, using (4) and (7),

$$(8) \quad ||F(x^0) + \underline{\varepsilon}(x, x^0)|| ||x - x^0|| < \frac{||x - x^0||}{2M} + \frac{||x - x^0||}{2M} = \frac{||x - x^0||}{M}.$$

Now, using (8), and recalling that, for $x^0 \in B_\delta(x^*)$, $||J_F^{-1}(x^0)|| \leq M$,

it follows that, for $x \in S_{v/2}(x^*)$, $x^0 \in B_\rho(x^*)$, the norm of the left side of equality (3) is less than

$$||x - x^0||.$$

But the norm of the right side of equality (3), for all $x \neq x^0$, is

$$||\eta - 1|| ||x - x^0|| > ||x - x^0||$$

since $\eta < 0$. Thus, equality (3) cannot hold if $x \in S_{v/2}(x^*)$ and

$x^0 \in B_\rho(x^*)$. In other words, taking $C = B_{v/2}(x^*)$, and taking $x^0 \in B_\rho(x^*)$,

since (2) \Rightarrow (3),

$$J_F^{-1}(x^0) F(x) \neq \eta(x - x^0)$$

whenever $\eta < 0$ and $x \in \dot{C}$, or

$$H_{x^0}(x) \neq \lambda x + (1 - \lambda) x^0$$

whenever $\lambda > 1$ and $x \in \dot{C}$. #

As an illustration, we show that the conclusion of the theorem is satisfied for the earlier example

$$G_1(x_1, x_2) = 1$$

$$G_2(x_1, x_2) = 2x_2 - 1.$$

In this case, for all $x \in \mathbb{R}^n$

$$J_G(x) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad I - J_G(x) = (I - J_G(x))^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Clearly the point $(1, 1)$ is a regular fixed point of G . Also note that for all x

$$H_{x^0}(x) = x - (I - J_G(x^0))^{-1} (x - G(x)) = (1, 1).$$

Now let x^0 be arbitrary and let C denote any open sphere containing x^0 and for which $(1, 1) \in \bar{C}$. Then $H_{x^0}(x) = (1, 1) \neq x^0 + \lambda(x - x^0)$ whenever $\lambda > 1$ and $x \in \dot{C}$, as in the conclusion to Theorem 1. For this example, although the hypotheses of Yamamuro's theorem, applied to G , do not hold, they are satisfied when applied to H . Moreover, Theorem 1 shows that any conditions which guarantee the existence of a regular fixed point also imply the assumptions of Lemma 1. In this sense, then, the assumptions of Lemma 1 are as weak as possible for the existence of a regular fixed point.

REFERENCE

- [1] Yamamuro, S. [1963]. "Some Fixed Point Theorems in Locally Convex Linear Spaces," Yokahama Math J. 11, 5-12.